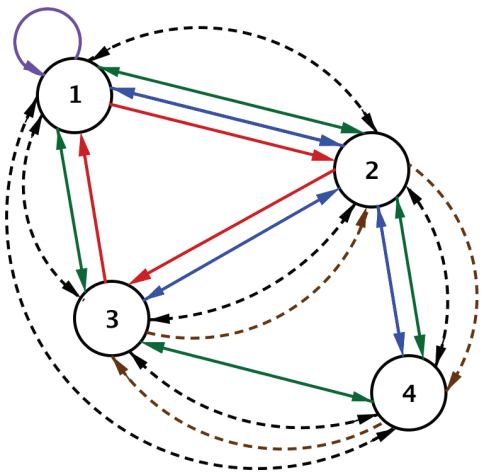
LESSON 4

Understanding Matrix Multiplication

Bus —
Subway - - - -

Opening Exercise

The subway and bus line network connecting four cities that we used in Lesson 2 is shown at the right. The bus routes connecting the cities are represented by solid lines, and the subway routes are represented by dashed lines.



1. Suppose we want to travel from City 2 to City 1, first by bus and then by subway, with no more than one connecting stop.

A. Complete the chart below showing the number of ways to travel from City 2 to City 1 using first a bus and then the subway. The first row has been completed for you.

First Leg (BUS)		Second Leg (SUBWAY)		Total Ways to Travel
City 2 to City 1:	2	City 1 to City 1:	0	$2 \cdot 0 = 0$
City 2 to City 2:	0	City 2 to City 1:	1	$0 \cdot 1 = 0$
City 2 to City 3:	2	City 3 to City 1:	1	$2 \cdot 1 = 2$
City 2 to City 4:	2	City 4 to City 1:	1	$2 \cdot 1 = 2$

total = 4

$P_{2,1}$

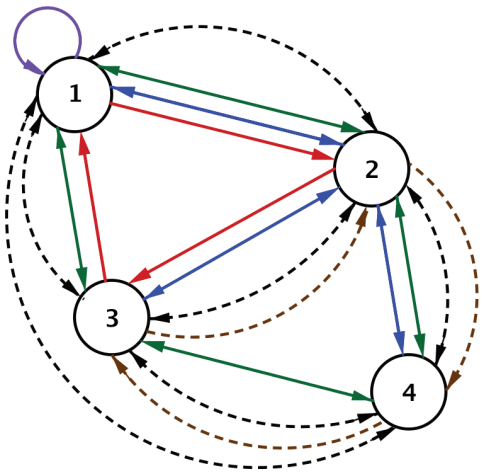
B. How many ways are there to travel from City 2 to City 1, first on a bus and then on a subway? How do you know?

4 ways

C. Why are the total ways to travel between City 2 to City 1 through City 1 equal to 0?

There's no subway route in City 1.

Exploratory Challenge: The Meaning of Matrix Multiplication



Suppose we want to travel between all cities, traveling first by bus and then by subway, with no more than one connecting stop.

2. Use a chart like the one in the Opening Exercise to help you determine the total number of ways to travel from City 1 to City 4 using first a bus and then the subway.

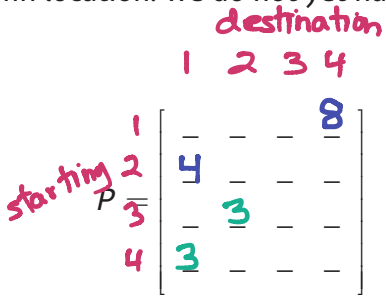
First Leg (BUS)		Second Leg (SUBWAY)		Total Ways to Travel
1 → 1	1	1 → 4	1	1 · 1 = 1
1 → 2	3	2 → 4	2	3 · 2 = 6
1 → 3	1	3 → 4	1	1 · 1 = 1
1 → 4	0	4 → 4	0	0 · 0 = 0

The total number of ways to travel from City 1 to City 4 by bus and then by subway: 8

$P_{1,4}$

3. Suppose we create a new matrix *P* to show the number of ways to travel between the cities, first by bus and then by subway, with no more than one connecting stop.

   A. Record your answers to Opening Exercises, Part B and Exercise 2 in the matrix below in the appropriate row and column location. We do not yet have enough information to complete the entire matrix.



   B. Explain how you decided where to record these numbers in the matrix.

The rows represent the starting city (1)
The columns represent the destination city. (4)

$P_{3,2}$

4. What is the total number of ways to travel from City 3 to City 2 first by bus and then by subway with no more than one connecting stop? Explain how you got your answer and where you would record it in matrix *P*.

Bus

$3 \to 1 = 2$
$3 \to 2 = 1$
$3 \to 3 = 0$
$3 \to 4 = 1$

Subway

$1 \to 2 = 1$
$2 \to 2 = 0$
$3 \to 2 = 2$
$4 \to 2 = 1$

$2 \times 1 = 2$
$1 \times 0 = 0$
$0 \times 2 = 0$
$1 \times 1 = 1$

$2 + 0 + 0 + 1 = \boxed{3}$

$P_{3,2}$

Matrix B, below, shows the number of bus lines connecting the cities in this transportation network, and matrix S, represents the number of subway lines connecting the cities in this transportation network.

$$B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

$P_{1,4}$

5. What does the product $b_{1,2}\, s_{2,4}$ represent in this situation? What is the value of this product?

Going from 1 to 2 by bus then 3
2 to 4 by subway. 2
$3 \times 2 = 6$

6. What does $b_{1,4}\, s_{4,4}$ represent in this situation? What is the value of this product? Does this make sense?

Going from 1 to 4 by bus then 0
4 to 4 by subway 0
$0 \times 0 = 0$

7. Calculate the value of the expression $b_{1,1}\, s_{1,4} + b_{1,2}\, s_{2,4} + b_{1,3}\, s_{3,4} + b_{1,4}\, s_{4,4}$. What is the meaning of this expression in this situation?

$1 + 6 + 1 + 0 = 8$ $P_{1,4}$

The total ways of traveling from 1 to 4 first by bus then by subway.

8. Circle the first row of B and the fourth column of S. How are these entries related to the expression above and your work in Exercise 2?

Aha!

9. A. Write an expression that represents the total number of ways you can travel between City 2 and City 1, first by bus and then by subway, with no more than one connecting stop.

$b_{2,1} \cdot s_{1,1} + b_{2,2} \cdot s_{2,1} + b_{2,3} \cdot s_{3,1} + b_{2,4} \cdot s_{4,1}$

$(2 \cdot 0) + (0 \cdot 1) + (2 \cdot 1) + (2 \cdot 1)$

$0 + 0 + 2 + 2 = \boxed{4}$

B. What is the value of this expression?

$p_{2,1}$

C. What is the meaning of the result?

D. Where in matrix *P* would you put this value?

10. A. Write an expression that represents the total number of ways you can travel between City 4 and City 1, first by bus and then by subway, with no more than one connecting stop.

$b_{4,1} \cdot s_{1,1} + b_{4,2} \cdot s_{2,1} + b_{4,3} \cdot s_{3,1} + b_{4,4} \cdot s_{4,1}$

$(0 \cdot 0) + (2 \cdot 1) + (1 \cdot 1) + (0 \cdot 1)$

$2 + 1$

B. What is the value of this expression? $\boxed{3} \leftarrow p_{4,1}$

C. What is the meaning of the result?

There are 3 ways to travel from City 4 to 1 first by bus then by Subway.

D. Where in matrix *P* would you put this value?

Discussion

11. A. What does each element of matrix *P* represent?

B. What patterns do you notice in the expressions in Exercises 7, 9 and 10?

C. Complete the sentence:

To calculate the element of matrix *P* in the 2nd row and 4th column, you would . . .

D. Complete the sentence:

The element of matrix *P* in the 4th row and 2nd column represents the number of way to travel . . .

E. Describe how to calculate any element in matrix *P*.

12. Complete matrix P that represents the routes connecting the four cities if you travel first by bus and then by subway.

$$P = \begin{bmatrix} 4 & 3 & 4 & 8 \\ 4 & 8 & 6 & 4 \\ 2 & 3 & 5 & 4 \\ 3 & 2 & 2 & 5 \end{bmatrix}$$

*Box method

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1\cdot5+2\cdot7 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Ex

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 4 & -2 \\ 0 & 7 \end{bmatrix}$$

$2\times3 \quad 3\times2$

$\hookrightarrow 2\times2$

$$\begin{bmatrix} -1 & 5 \\ 4 & -2 \\ 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & -2 \end{bmatrix} \quad \begin{bmatrix} -2+12+0 & 10-6+7 \\ 1+0+0 & -5+0-14 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 11 \\ 1 & -19 \end{bmatrix}$$

Lesson Summary

Operation	Symbols	How to Calculate	Example
Matrix Multiplication	$A \cdot B$	To find the element in the ith row and jth column of the product matrix, multiply corresponding elements from the ith row of the first matrix by the jth column of the second matrix, and then add the results. Repeat this process for each element in the product matrix.	$\begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & -1 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 0 \end{bmatrix} =$

When we multiply two matrices together, such as an $m \times n$ matrix by an $n \times p$ matrix, what is the size of the resulting matrix?

Size of final product

$(m \times n) \cdot (n \times p)$

Inner numbers match

The resulting matrix has size _____.

$(2\times2)(2\times2) = (2\times2)$

$2\times3 \quad 3\times1 = 2\times1$

rows × columns

NAME: _____ PERIOD: _____ DATE: _____

Homework Problem Set

1. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ represent the bus routes of two companies between two cities. Find the product $A \cdot B$, and explain the meaning of the entry in row 1, column 2 of $A \cdot B$ in the context of this scenario.

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2. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ represent the bus routes of two companies between three cities.

   A. Let $C = A \cdot B$. Find matrix C, and explain the meaning of entry $c_{1,3}$.

   B. Nina wants to travel from City 3 to City 1 and back home to City 3 by taking a direct bus from Company A on the way to City 1 and a bus from Company B on the way back home to City 3. How many different ways are there for her to make this trip?

   C. Oliver wants to travel from City 2 to City 3 by taking first a bus from Company A and then taking a bus from Company B. How many different ways can he do this?

   D. How many routes can Oliver choose from if travels from City 2 to City 3 by first taking a bus from Company B and then taking a bus from Company A?

3. Consider the matrices

$$A = \begin{bmatrix} 3 & 1 & -\frac{1}{2} \\ 2 & \frac{2}{3} & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Multiply AB and BA or explain why you cannot.

For the matrices given below, perform each of the following calculations or explain why the calculation is not possible.

$$A = \begin{bmatrix} \frac{1}{2} & 3 \\ 2 & \frac{2}{3} \end{bmatrix} \qquad B = \begin{bmatrix} 9 & -1 & 2 \\ -3 & 4 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 3 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & -2 & \frac{1}{2} \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

4. AB	5. BC
6. AC	7. AD
8. A^2	9. C^2
10. $2A + B$	11. $B + BC$

12. Let F be an $m \times n$ matrix. Then what do you know about the dimensions of matrix G in the problems below if each expression has a value?

    A. $F + G$

    B. FG

    C. GF

13. Consider an $m \times n$ matrix A such that $m \neq n$. Explain why you cannot evaluate A^2.

14. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ represent the routes of three airlines A, B, and C between three cities.

A. Zane wants to fly from City 1 to City 3 by taking Airline A first and then Airline B second. How many different ways are there for him to travel?

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B. Zane did not like Airline A after the trip to City 3, so on the way home, Zane decides to fly Airline C first and then Airline B second. How many different ways are there for him to travel?