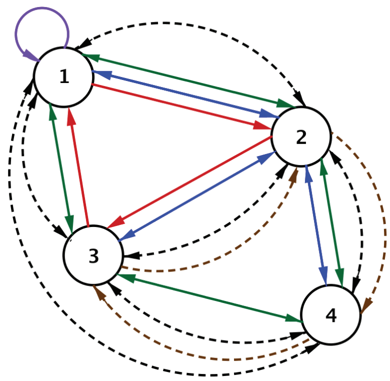
LESSON 5 Properties of Matrices

Opening Exercise



In Lesson 4, we used the subway and bus line network connecting four cities. The bus routes connecting the cities are represented by solid lines, and the subway routes are represented by dashed lines.

Matrix B, below, shows the number of bus lines connecting the cities in this transportation network, and matrix S, represents the number of subway lines connecting the cities in this transportation network.

$$B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}$$

We found that $B \cdot S = P$ as shown below.

$$B \cdot S = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{4} & \mathbf{8} \\ \mathbf{4} & \mathbf{8} & \mathbf{6} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} & \mathbf{5} & \mathbf{4} \\ \mathbf{3} & \mathbf{2} & \mathbf{2} & \mathbf{5} \end{bmatrix} = P$$

1. Calculate matrix M that represents the routes connecting the four cities if you travel first by subway and then by bus.

$$\underset{S}{\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix}} \cdot \underset{B}{\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} 4 & 3 & 3 & 3 \\ 3 & 8 & 3 & 1 \\ 5 & 5 & 6 & 4 \\ 7 & 5 & 3 & 4 \end{bmatrix}$$

2. Should these two matrices (P and M) be the same? Explain your reasoning.

No, because rows x columns are now diff.

$B \cdot S \neq S \cdot B$ (not commutative)

We've shown that matrix multiplication is generally not commutative, meaning that as a general rule for two matrices A and B, $A \cdot B \neq B \cdot A$.

3. Explain why $F \cdot G = G \cdot F$ in each of the following examples.

A. $F = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 2 & 6 \\ 4 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ (2F)

$F \cdot G$

$F \cdot 2F$

$2F \cdot F \rightarrow G \cdot F$

B. $F = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $F \cdot G = G \cdot F$

$F \cdot 0$ zero multiplication

$0 \cdot F$

C. $F = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $F \cdot G = G \cdot F$

Identity matrix $= I$

$1 \cdot 5 = 5$

$F \cdot I = F$

$I \cdot F = F$

D. $F = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2 \end{bmatrix}$, $G = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$F \cdot G$

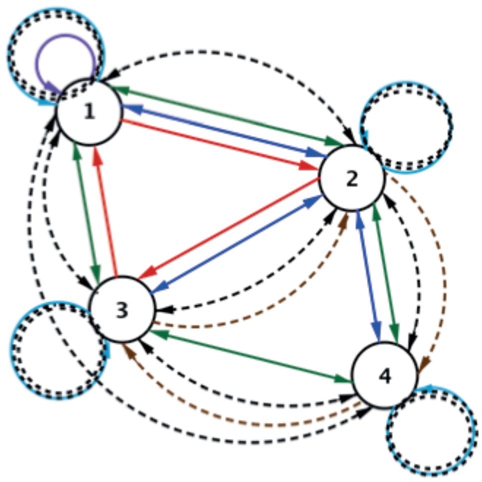
$3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$3I$

$F \cdot 3I$

$3(F \cdot I)$

$\underline{3}\, I \cdot F \rightarrow G \cdot F$



Suppose each city had a trolley car that ran a route between tourist destinations. The double dotted loops represent the trolley car routes. Remember that straight lines indicate bus routes, and dotted lines indicate subway routes.

4. Explain why the matrix I shown below would represent the number of routes connecting cities by trolley car in this transportation network.

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The ones represent each trolley route.

5. Recall that B is the bus route matrix. Show that $I \cdot B = B$. Explain why this makes sense in terms of the transportation network.

$$B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

6. The real number 1 has the property that $1 \cdot a = a$ for all real numbers a, and we call 1 the *multiplicative identity.* Why would mathematicians call I an identity matrix?

It works the same way as 1 in real number.

7. What would be the form of a 2×2 identity matrix? What about a 3×3 identity matrix?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$

   A. Construct a matrix Z such that $A + Z = A$. Explain how you got your answer.

   $$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   B. Explain why $k \cdot Z = Z$ for any real number k.

   $$k \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   C. The real number 0 has the properties that $a + 0 = a$ and $a \cdot 0 = 0$ for all real numbers a. Why would mathematicians call Z a zero matrix?

   The zero matrix has the same properties as 0.

9. In this lesson you learned that the commutative property does not hold for matrix multiplication. This exercise asks you to consider other properties of real numbers applied to matrix arithmetic.

   A. Is matrix addition associative? That is, does $(A + B) + C = A + (B + C)$ for matrices A, B, and C that have the same dimensions? Explain your reasoning.

   True

   Adding real numbers apply the same rule.

B. Is matrix multiplication associative? That is, does $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ for matrices A, B, and C for which the multiplication is defined? Explain your reasoning.

$$\left(\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}}\underset{B}{\begin{bmatrix} -1 & 0 \\ -3 & 5 \end{bmatrix}}\right) \cdot \underset{C}{\begin{bmatrix} 4 & 8 \\ 0 & -1 \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}}\left(\underset{B}{\begin{bmatrix} -1 & 0 \\ -3 & 5 \end{bmatrix}}\underset{C}{\begin{bmatrix} 4 & 8 \\ 0 & -1 \end{bmatrix}}\right)$$

C. Is matrix addition commutative? That is, does $A + B = B + A$ for matrices A and B with the same dimensions?

Yes.

Lesson Summary

Identity Matrix: The $n \times n$ *identity matrix* is the matrix whose entry in row i and column i is 1, and all other entries are all zero.

The identity matrix is denoted by I.

The 2×2 identity matrix is $\begin{bmatrix} \_\_ & 0 \\ \_\_ & \_\_ \end{bmatrix}$, and the 3×3 identity matrix is $\begin{bmatrix} 1 & 0 & \_\_ \\ \_\_ & \_\_ & 0 \\ 0 & \_\_ & 1 \end{bmatrix}$.

If the size of the identity matrix is not explicitly stated, then the size is implied by context.

Zero Matrix: The $m \times n$ *zero matrix* is the $m \times n$ matrix in which all entries are equal to zero.

For example, the 2×2 zero matrix is $\begin{bmatrix} \_\_ & 0 \\ \_\_ & \_\_ \end{bmatrix}$, and

the 3×3 zero matrix is $\begin{bmatrix} \_\_ & 0 & \_\_ \\ \_\_ & \_\_ & 0 \\ 0 & \_\_ & 0 \end{bmatrix}$.

If the size of the zero matrix is not specified explicitly, then the size is implied by context.

NAME: ______________________ PERIOD: ____________________ DATE: _______________

Homework Problem Set

1. Consider the matrices.

$$A = \begin{bmatrix} 3 & 1 & -\frac{1}{2} \\ 2 & \frac{2}{3} & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   A. Would you consider B to be an identity matrix for A? Why or why not?

   B. Would you consider $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ an identity matrix for A? Why or why not?

2. Recall the bus and trolley matrices from the lesson:

$$B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   A. Explain why it makes sense that $BI = IB$ in the context of the problem.

   B. Multiply out BI and show $BI = IB$.

   C. Consider the multiplication that you did in Part B. What about the arrangement of the entries in the identity matrix causes $BI = B$?

3. Let $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0 \end{bmatrix}$ represent airline flights of one airline between 4 cities.

A. We use the notation A^2 to represent the product $A \cdot A$. Calculate A^2. What do the entries in matrix A^2 represent?

B. Jade wants to fly from City 1 to City 4 with exactly one stop. How many different ways are there for her to travel?

C. Now Jade wants to fly from City 1 to City 4 with exactly two stops. How many different ways are there for her to choose?