LESSON 10 Recursive Formulas for Sequences

LEARNING OBJECTIVES

- Today I am: determining the rule for the Fibonacci sequence.
- So that I can: learn to write recursive formulas.
- I'll know I have it when I can: write a recursive formula given an explicit formula.

Opening Exercise

One of the most famous sequences is the *Fibonacci* sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The number of petals on a flower is often a Fibonacci number.

1 3 5 13

Why are four-leaf clovers so rare? Because 4 isn't a Fibonacci number!

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1. How is each term of the Fibonacci sequence generated?

Add the two preceding terms.

$n-2$, $n-1$, n, $n+1$, $n+2$

In looking for patterns in sequences, it is useful to look for a pattern in how each term relates to the previous term. If there is a consistent pattern in how each term relates to the previous one, it is convenient to express this pattern using a ***recursive*** definition for the sequence that gives the first term and a formula for how the n^{th} term relates to the $(n-1)$ term.

previous term.

To describe Fibonacci's sequence with an explicit formula would be difficult. Instead, we'll use a *recursive* formula. A recursive formula relates a term in the sequence to the preceding term or terms of the sequence.

1, 1, 2, 3, 5, 8, 13, 21, 34, ...

	Recursive Formula	Starting Conditions
Function Notation	$f(n+1) = f(n) + f(n-1)$	where $f(1) = 1$, $f(2) = 1$, and $n \geq 2$
"*a* sub" Notation	$a_{n+1} = a_n + a_{n-1}$	where $a_1 = 1$, $a_2 = 1$, and $n \geq 2$

The next term in the sequence.

The current term in the sequence.

The previous term in the sequence.

Notice that in the Fibonacci sequence, each term depends on the **two** previous terms. This means we had to know the first two terms in order to start the sequence. Most sequences you'll use only rely on the previous term.

2. When writing a recursive formula, what piece of information is necessary to include along with the formula?

→ The first term $f(1)$ or a_1

There is no hard-and-fast requirement that all recursive sequences start with the index at 1. In some cases, it is convenient to start the index at 0. However, in this module, we mostly stay with sequences starting at index 1.

$n \geq 1$ $f(0)$

Practice with Recursive Formulas

For each sequence below, match it to its recursive formula and then write in the a_1 term.

3. 1, 0, −1, −2, −3, . . . C
   −1
4. 5, 10, 15, 20, 25, . . . A
   +5
5. 400, 200, 100, 50, . . . E
6. 32, 52, 72, 92, 112, . . . B
7. −1, 2, −4, 8, −16, . . . D

A. $a_n = a_{n-1} + \boxed{5}$ $a_1 =$ 5

B. $a_n = a_{n-1} + \boxed{20}$ $a_1 =$ 32

C. $a_n = a_{n-1} \boxed{-1}$ $a_1 =$ 1

D. $a_n = \boxed{-2}a_{n-1}$ $a_1 =$ −1

E. $a_n = \boxed{\frac{1}{2}} \cdot a_{n-1}$ $a_1 =$ 400

Recursive versus Explicit Formulas

8. Consider Akelia's sequence 5, 8, 11, 14, 17, ...

   A. What is the next number in the sequence? 20 $d=3$

   When asked to find a formula for this sequence, Akelia wrote the following on a piece of paper.

$$a_1 = 5$$
$$a_2 = 8 = 5 + 3$$
$$a_3 = 11 = 5 + 3 + 3 = 5 + 2 \cdot 3$$
$$a_4 = 14 = 5 + 3 + 3 + 3 = 5 + 3 \cdot 3$$

   B. Use her reasoning to write an explicit formula for Akelia's sequence.

$a_1 = 5$ $\quad a_n = a_1 + d(n-1)$

$d = 3$ $\quad a_n = 5 + 3(n-1)$

$\quad = 5 + 3n - 3$

$\boxed{a_n = 3n+2}$

   C. Explain how each part of the formula relates to the sequence.

9. When Johnny saw the sequence 5, 8, 11, 14, 17, ..., he wrote the following recursive formula:

$$A(n+1) = A(n) + 3 \text{ for } \boxed{n \geq 1} \text{ and } A(1) = 5.$$

first term

starting term

Explain Johnny's formula.

the next term

the current term

common difference.

10. Why does Akelia's formula have a "times 3" in it, while Johnny's formula has a "plus 3"?

11. If we wanted the 200th term of the sequence, which formula would be more useful?

Explicit.

$a_{200} = 3(200) + 2$

$= 602$

12. If we wanted to know how the sequence changes from one term to the next, which formula would be more useful?

Recursive.

$a_{n+1} = a_n \boxed{+ 3}$

13. Using Johnny's recursive formula, what would we need to know if we wanted to find the 200th term?

The previous term.

199th.

14. Akelia asked Johnny: "What would happen if we change the '+' sign in your formula to a '−' sign? To a '•' sign? To a '÷' sign?"

    A. What sequence does $A(n + 1) = A(n) - 3$ for $n \geq 1$ and $A(1) = 5$ generate?

    5, 2, −1, −4, −7, ...

    B. What sequence does $A(n + 1) = A(n) \cdot 3$ for $n \geq 1$ and $A(1) = 5$ generate?

    5, 15, 45, 135, 405, ...

    C. What sequence does $A(n + 1) = A(n) \div 3$ for $n \geq 1$ and $A(1) = 5$ generate?

    $5, \frac{5}{3}, \frac{5}{9}, \frac{5}{27}, \frac{5}{81}, \ldots$

15. Ben made up a recursive formula and used it to generate a sequence. He used $B(n)$ to stand for the n^{th} term of his recursive sequence.

    A. What does $B(3)$ mean?

    The third term

    B. What does $B(k)$ mean?

    The k^{th} term

    C. If $B(n + 1) = 33$ and $B(n) = 28$, write a possible recursive formula involving $B(n + 1)$ and $B(n)$ that would generate 28 and 33 in the sequence.

    ___, ___, $\underset{B(n)}{28}$, $\underset{B(n+1)}{33}$, ___

    $B(n+1) = B(n) + 5$

    D. What does $2B(7) + 6$ mean?

    E. What does $B(n) + B(k)$ mean? Would it necessarily be the same as $B(n + k)$? Explain.

    F. What does $B(17) - B(16)$ mean?

Write a recursive formula for each sequence below. The explicit formulas are included to help you see the connection between an explicit formula and its corresponding recursive formulas.

	Sequence	Explicit Formula	Recursive Formula	Starting Conditions ($n \ge 1$)
16.	+3 +3 $-4, -1, 2, 5, \ldots$ a	$f(n) = 3n - 7$	$f(n+1) = f(n) + 3$	$f(1) = -4$, $n \ge 1$
17.	+3 $3, 6, 9, 12, \ldots$ a	$f(n) = 3n$	$f(n+1) = f(n) + 3$	$f(1) = 3$, $n \ge 1$
18.	×2 ×2 $2, 4, 8, 16, \ldots$ g	$f(n) = 2^n$	$f(n+1) = 2 \cdot f(n)$	$f(1) = 2$, $n \ge 1$
19.	$3, 9, 27, 81, \ldots$ g	$f(n) = 3^n$	$f(n+1) = 3 \cdot f(n)$	$f(1) = 3$, $n \ge 1$
20.	$-1, -2, -3, -4, \ldots$ a -1 -1 -1	$f(n) = -n$	$f(n+1) = f(n) - 1$	$f(1) = -1$, $n \ge 1$

Lesson Summary

Recursive sequence: A *recursive sequence* is a sequence that

(1) is defined by specifying the values of one or more *initial terms* and

(2) has the property that the remaining terms satisfy a recursive formula that describes the value of a term based upon an expression in numbers, previous terms, or the index of the term.

Example: Consider a sequence given by the formula $a_n = a_{n-1} - 5$, where $a_1 = 12$ and $n \geq 2$.

List the first five terms of the sequence.

$a_1 = 12$

$a_2 = a_1 - 5 = 12 - 5 = 7$

$a_3 = a_2 - 5 = 7 - 5 = 2$

$a_4 = a_3 - 5 = 2 - 5 = -3$

$a_5 = a_4 - 5 = -3 - 5 = -8$

The sequence starts with 12, 7, 2, −3, −8.

NAME: ______________________ PERIOD: __________ DATE: __________

Homework Problem Set

For Problems 1–4, list the first five terms of each sequence.

1. $a_n = a_{n-1} + 6$, where $a_1 = 11$ for $n \geq 1$

2. $a_n = a_{n-1} \div 2$, where $a_1 = 50$ for $n \geq 1$

3. $f(n) = -2f(n-1) + 8$ and $f(1) = 1$ for $n \geq 1$

4. $f(n) = f(n-1) + n$ and $f(1) = 4$ for $n \geq 1$

For Problems 5–10, write a recursive formula for each sequence given or described below.

5. It follows a plus one pattern: 8, 9, 10, 11, 12, ...

6. It follows a multiply by 10 pattern: 4, 40, 400, 4000, ...

7. It has a general formula of $f(n) = -3n + 2$ for $n \geq 1$.

8. It has a general formula of $f(n) = -1(12)^{n-1}$ for $n \geq 1$.

9. Doug accepts a job where his starting salary is \$30,000 per year, and each year he receives a raise of \$3,000.

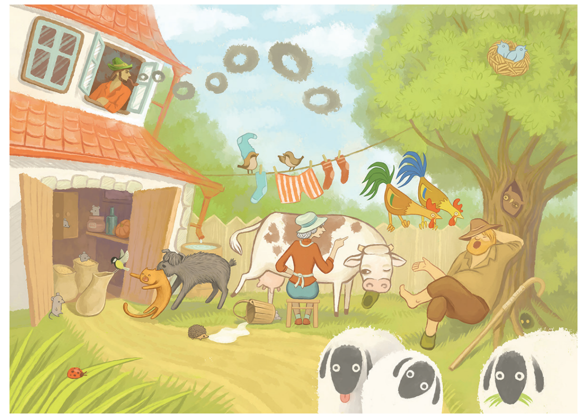
10. A bacteria culture has an initial population of 10 bacteria, and each hour the population triples in size.

11. Each sequence below gives an explicit formula. Write the first five terms of each sequence. Then, write a recursive formula for the sequence.

    A. $a_n = 2n + 10$ for $n \geq 1$

    B. $f(n) = \left(\frac{1}{2}\right)^{n-1}$ for $n \geq 1$

12. The nursery rhyme "This is the House that Jack Built" can be considered a recursive poem or story. How is it recursive? What are the beginning conditions? What happens at each stage of the story?



This is the house that
Jack built.

This is the malt
That lay in the house that
Jack built.

This is the rat,
That ate the malt
That lay in the house that
Jack built.

This is the cat,
That killed the rat,
That ate the malt
That lay in the house that
Jack built.

This is the dog,
That worried the cat,
That killed the rat,
That ate the malt
That lay in the house that
Jack built.

This is the cow with the
crumpled horn,
That tossed the dog,
That worried the cat,
That killed the rat,
That ate the malt
That lay in the house that
Jack built . . .