LESSON 7

The "Perfect" Rectangle

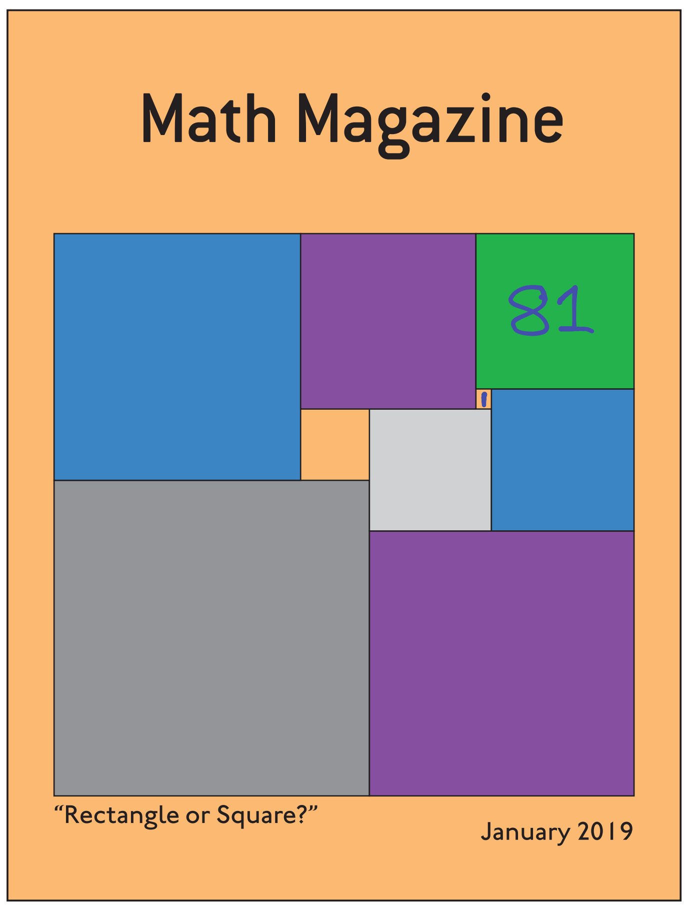
LEARNING OBJECTIVES

- Today I am: examining the cover picture of a math magazine.
- So that I can: identify square numbers and their square roots.
- I'll know I have it when I can: simplify an expression like $\sqrt[3]{216ab^6}$.

Exploratory Activity

1. What questions do you have about this magazine cover?

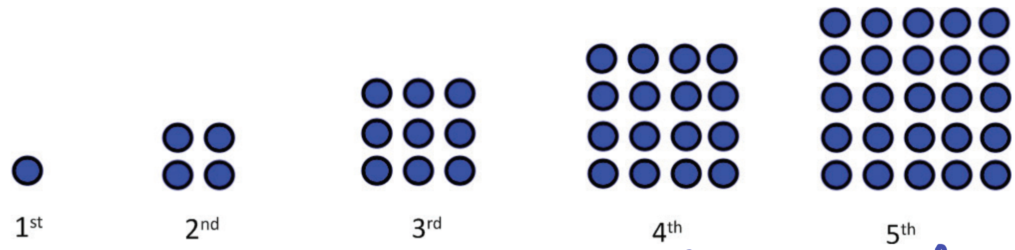
2. The top right square is 81 square units and the smallest square is 1 square unit. The rest of the figure is made of squares. What is the area of the entire figure? Is it a square or a rectangle?



Other Sequences

In the Lesson 5 Homework Problem Set you extended the patterns for several different non-number sequences. Often these sequences can be better understood as numerical sequences.

The sequence of *perfect squares* {1, 4, 9, 16, 25, ...} earned its name because the ancient Greeks realized these quantities could be arranged to form square shapes.



1^2 2^2 3^2 4^2 5^2

3. If $S(n)$ denotes the n^{th} square number, what is a formula for $S(n)$?

$S(n) = n^2$ for $n = 1, 2, 3, 4, \dots$

4. Prove whether or not 169 is a perfect square.

Yes because $13 \times 13 = 169$

5. Prove whether or not 200 is a perfect square.

No

Perfect square

1, 4, 9, 16, 25, 36, 49, 64, 81,
100, 121, 144, 169, 196, 225

$25^2 = 625$

6. If $S(n) = 225$, then what is n?

$n^2 = 225$
$n = 15$

7. Which term is the number 400 in the sequence of perfect squares? How do you know?

20^{th} term

> "In the 9th century, Arab writers usually called one of the equal factors of a number *jadhr* ("root"), and their medieval European translators used the Latin word *radix* (from which derives the adjective *radical*)."
>
> -ENCYCLOPÆDIA BRITANNICA

radical = $\sqrt{}$

The "equal factors of a number" means $a \cdot a$ and the "root" of this expression is a. Instead of writing out all the words, mathematicians came up with a "radical" symbol, $\sqrt{}$, to signify the root. So $\sqrt{a \cdot a} = a$, as long as a is a positive number.

$\sqrt{3\cdot 3} = 3 = \sqrt{9} = \sqrt{3^2}$ $\sqrt{225} = \sqrt{15^2} = \sqrt{15\times 15} = 15$

There came a need to solve problems such as $x^3 = 8$. In this case, we need the third root or cube root of 8. This is written $x = \sqrt[3]{8}$. Since $8 = 2 \cdot 2 \cdot 2$, we get $x = \sqrt[3]{8} = 2$.

$3\sqrt{8} \neq \sqrt[3]{8}$

In general, we say the n^{th} root of a or $\sqrt[n]{a}$ or $a^{\frac{1}{n}}$.

$x^3 = 8$ $x^2 = 9$

$\sqrt{x} = x^{1/2}$ $\sqrt[3]{x^3} = \sqrt[3]{8}$ $\sqrt{x^2} = \sqrt{9}$

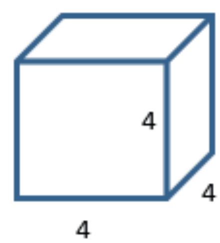
$\sqrt[3]{x} = x^{1/3}$ $x = 2$ $x = 3$

Focus on the Reading

8. From the reading, we see that $\sqrt[3]{-27} = -3$. What is $\sqrt[3]{27}$? $= 3$

$\sqrt{-9} = DNE$ $(-3)(-3)(-3)$

$(-3)(-3) = 9$



9. The diagram at the right shows how cubed numbers can be represented in geometric terms. What is the volume of the cube?

$4^3 = 64$

Determine the following roots.

10. $\sqrt{49}$ $\sqrt{7^2} = 7$

11. $\sqrt[3]{1}$ $= 1$

12. $\sqrt[3]{-8}$ $= -2$

13. $\sqrt{16}$ $\sqrt{4^2} = 4$

?

14. $\sqrt[4]{64}$ $8 \cdot 8$ $2\cdot 2\cdot 2 \; 2\cdot 2\cdot 2$

15. $\sqrt{100}$ $\sqrt{10^2} = 10$

16. $\sqrt{36}$ $\sqrt{6^2} = 6$

17. $\sqrt[3]{-1}$ $= -1$ $\sqrt[3]{(-1)^3}$

All of the numbers in the radicals in Exercises 10–17 were either perfect squares or perfect cubes. Let's look at numbers that are not so "perfect".

To better understand non-perfect numbers and how to find their roots, we'll first look at how numbers can be broken down.

$\sqrt{20} = ?$

$2\cdot 2\cdot 5$

Prime numbers can only be evenly divided by themselves and 1.

You will need: a highlighter

18. You will be highlighting the 100-chart below using specific rules. Highlight all the *prime numbers*. Leave all of the composite numbers alone. Then circle the one number that is neither prime nor composite.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

19. What is special about prime numbers (the ones highlighted in your 100-chart)?

20. Choose any four composite numbers from your 100-chart and break them down into their prime factors.

For example: $720 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$.

$80 = 8 \times 10$

$= 2 \times 2 \times 2 \times 2 \times 5$

$\sqrt{80} = ?$

$72 = 2 \times 2 \times 2 \times 3 \times 3$

$15 = 3 \times 5$

$92 = 2 \times 46$

$2 \times 2 \times 23$

$\sqrt{4}\sqrt{20}$

$2\sqrt{20}$

$\sqrt{4} = 2$

$\sqrt{81} = 9$

$\sqrt{20} = \sqrt{2 \times 2 \times 5}$

$\sqrt{4 \times 5}$

$= 2\sqrt{5}$

$\sqrt{80} = \sqrt{2 \times 2 \times 2 \times 2 \times 5}$

$= 2 \cdot 2\sqrt{5}$

$\sqrt{16}\sqrt{5} = 4\sqrt{5}$

$\sqrt{4}\sqrt{5}$ $\sqrt{16}$ $4\sqrt{5}$

 $2\sqrt{5}$

21. To find the square root of a number like 720, you need its prime factorization. Explain each step of this problem using the thought bubbles.

Example:

$$\sqrt{720} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5} = 2 \cdot 2 \cdot 3\sqrt{5} = 12\sqrt{5}$$

Find prime factorization

Pair up numbers Bring one out of the radical for each pair.

Mult. numbers outside radical

22. Use the prime factorization of the four numbers you chose in Exercise 20 to find the square root of each one.

$\sqrt{72} = \sqrt{2\times 2\times 2\times 3\times 3} = 2\cdot 3\sqrt{2}$

$\sqrt{36}\sqrt{2}$ $= 6\sqrt{2}$

$6\sqrt{2}$

$\sqrt{15} = \sqrt{3\times 5} = \sqrt{15}$

$\sqrt{92} = \sqrt{2\cdot 2\cdot 23} = 2\sqrt{23}$

$\sqrt{200} = \sqrt{10\cdot 10\cdot 2} = 10\sqrt{2}$

$2\cdot 2\cdot 2\cdot 5\cdot 5$

$\sqrt{50} = 5\sqrt{2}$

$\sqrt{x^3} = \sqrt{x\cdot x\cdot x} = x\sqrt{x}$

$\sqrt{x^5y^4} = \sqrt{x\cdot x\cdot x\cdot x\cdot x\cdot y\cdot y\cdot y\cdot y} = x^2y^2\sqrt{x}$

$\sqrt{x^7y^5z^2} = x^3y^2z\sqrt{xy}$

A similar procedure is used when you have variables under the radical sign. For example:

$\sqrt{x^4y^2} = x^2y$

$$\sqrt{720x^3y^2z} = \sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5\cdot x\cdot x\cdot x\cdot y\cdot y\cdot z} =$$

$x=-4$

$\sqrt[3]{x^5y^7} = xy^2\sqrt[3]{x^2y}$

$$\sqrt{2\cdot 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5\cdot x\cdot x\cdot x\cdot y\cdot y\cdot z} =$$

$$2\cdot 2\cdot 3\cdot x\cdot y\sqrt{5xz} = 12x|y|\sqrt{5xz}$$

$\sqrt{x^2} = x$

$\sqrt{(-4)^2} = -4$

$\sqrt{16} = -4$

$4 = |-4|$

23. **Challenge** Why are the absolute value bars needed on y but not on x?

Rules for absolute value:

1. Even root index: $\sqrt{\ }$, $\sqrt[4]{\ }$
2. Even power inside: x^2, x^4
3. Odd power outside

Ex: $\sqrt{x^4} = x^2$

$\sqrt{x^6} = |x^3|$

$\sqrt{4x^2} = 2|x|$

Use prime factorization to simplify radicals including those with variables.

24. $\sqrt{18x^3}$

$\sqrt{3\cdot 3\cdot 2x^3}$

$3x\sqrt{2x}$

25. $\sqrt{68xy^3}$

$2\cdot 2\cdot 17$

$2y\sqrt{17xy}$

26. $\sqrt{100y^4z^3}$

$10y^2z\sqrt{z}$

27. $\sqrt{36x}$

$6\sqrt{x}$

28. $\sqrt{92ab^3c^2}$

$2\cdot 2\cdot 23$

$2b|c|\sqrt{23ab}$

29. $\sqrt{24a^6b^4}$

$2\cdot 2\cdot 2\cdot 3$

$2|a^3|b^2\sqrt{6}$

So far we've focused on square roots using the radical symbol $\left(\sqrt{\ }\right)$, but there are other roots you can take of numbers. The mechanics are similar to what you do with square roots. For example, for a cube root $\left(\sqrt[3]{\ }\right)$ you need 3 of the same factor when you find the prime factorization.

So $\sqrt[3]{16} = \sqrt[3]{2\cdot 2\cdot 2\cdot 2} = 2\sqrt[3]{2}$

When you have simplified the radical as much as possible, the expression is in ***simplest radical form***.

30. Complete the table below showing examples of different roots.

Radical with Index	Index	Numeric Example	Algebraic Example
$\sqrt[3]{}$	3	$\sqrt[3]{8} =$	$\sqrt[3]{27x^5} =$ $3x\sqrt[3]{x^2}$ $3\cdot3\cdot3$
$\sqrt[4]{}$		$\sqrt[4]{81} =$	$\sqrt[4]{32x^4y^3} =$ $2\lvert x\rvert\sqrt[4]{2y^3}$ $2\cdot2\cdot2\cdot2\cdot2$
$\sqrt[5]{}$		$\sqrt[5]{1} =$	$\sqrt[5]{486x^5y^6} =$ $3xy\sqrt[5]{2y}$ $2\cdot3\cdot3\cdot3\cdot3\cdot3$

31. The square root symbol $(\sqrt{})$ could be written as $(\sqrt[2]{})$. Why do you think they leave off the ***index*** of 2?

$\sqrt{-4} = \text{DNE}$ $\sqrt[3]{-64} = -4$

$\downarrow$

$(-4)(-4)(-4)$

Simplify the radical expressions below.

32. $\sqrt{125x^3y^2}$ $5x\lvert y\rvert\sqrt{5x}$	33. $\sqrt[3]{216ab^6}$ $2\cdot108$ $2\cdot2\cdot54$ $2\cdot2\cdot2\cdot27$ $2\cdot2\cdot2\cdot3\cdot3\cdot3$ $6b^2\sqrt[3]{a}$	34. $\sqrt[4]{512a^5bc^4}$
35. $\sqrt[3]{200k^3m^5n^2}$ $2\cdot2\cdot2\cdot5\cdot5$ $2km\sqrt[3]{25m^2n^2}$	36. $\sqrt{80p^3}$ $4\cdot4\cdot5$ $2\cdot2\cdot2\cdot2\cdot5$ $4p\sqrt{5p}$	37. $\sqrt[5]{160d^2e^6}$

38. Darcy says $\sqrt{-8}$ is not possible but $\sqrt[3]{-8} = -2$. Explain Darcy’s thinking.

Simplify each expression below.

39. $\sqrt[3]{125}$

40. $\sqrt[3]{-125}$

41. $\sqrt[3]{1}$

42. $\sqrt[3]{-1}$

43. $\sqrt[5]{32}$

44. $\sqrt[5]{-32}$

45. $\sqrt[3]{27}$

46. $\sqrt[3]{-27}$

47. Complete the Lesson Summary.

Lesson Summary

Radical Rules

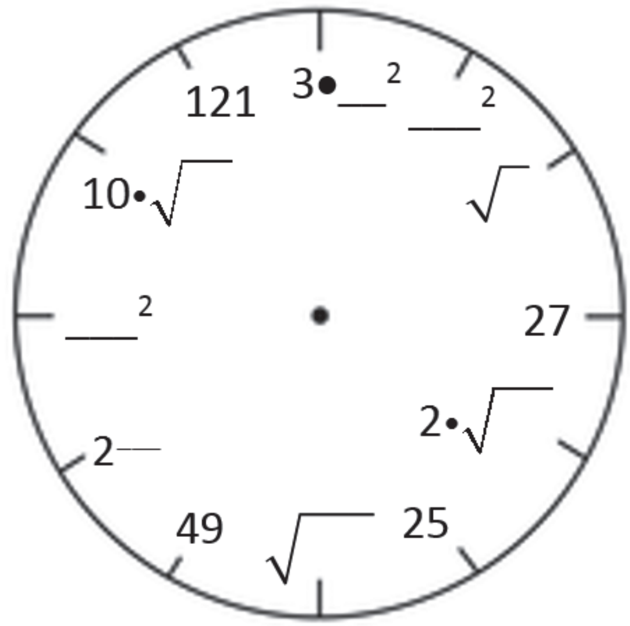
For real numbers $a \geq 0$ and $b \geq 0$

General Rule	Example
$\sqrt{ab} = \sqrt{\_\_\_\_} \cdot \sqrt{\_\_\_\_}$	$\sqrt{9 \cdot 4} = \sqrt{9} \cdot \sqrt{4} = \_\_\_\_$
$\sqrt[n]{ab} = \sqrt[n]{\_\_\_\_} \cdot \sqrt[n]{\_\_\_\_}$	$\sqrt[3]{8 \cdot 27} = \sqrt[3]{8} \cdot \sqrt[3]{27} = \_\_\_\_$
$\sqrt{(-a) \cdot (-a)} = \lvert\_\_\_\_\rvert$	$\sqrt{(-3) \cdot (-3)} = \lvert\_\_\_\_\rvert = \_\_\_\_$
$\sqrt{a \cdot a \cdot a} = \_\_\_\_\sqrt{\_\_\_\_}$	$\sqrt{3 \cdot 3 \cdot 3} = \_\_\_\_\sqrt{\_\_\_\_}$
$\sqrt[3]{a \cdot a \cdot a} = \_\_\_\_$	$\sqrt[3]{7 \cdot 7 \cdot 7} = \_\_\_\_$
$\sqrt[3]{a \cdot a \cdot a \cdot a} = \_\_\_\_\sqrt[3]{\_\_\_\_}$	$\sqrt[3]{2 \cdot 2 \cdot 2 \cdot 2} = \_\_\_\_\sqrt[3]{\_\_\_\_}$
$\sqrt[3]{a^3} = a$ and $\sqrt[3]{(-a)^3} = -a$	$\sqrt[3]{(-2)^3} = \_\_\_\_$
$\sqrt[5]{a^5} = a$ and $\sqrt[5]{(-a)^5} = -a$	$\sqrt[5]{6^5} = \_\_\_\_$

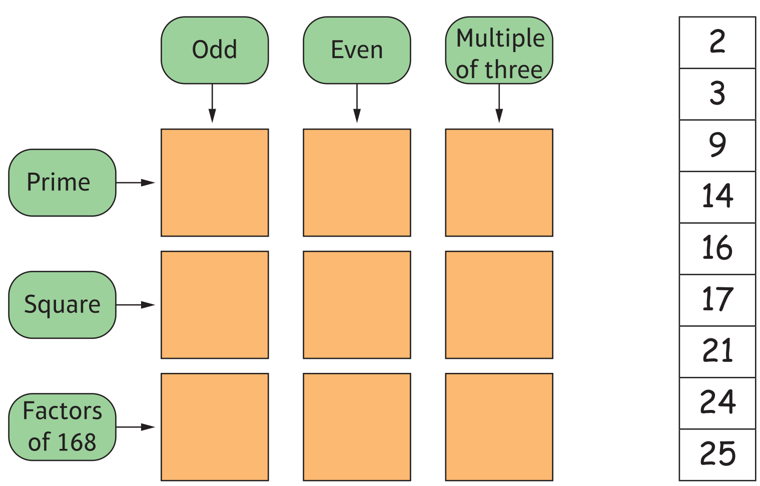
NAME: ______________________ PERIOD: ____________ DATE: ____________

Homework Problem Set

1. Beth bought a new clock but it was missing a few numbers and symbols. Draw in the symbols or write in the numbers needed to make this equivalent to a normal clock.



2. Use the numbers in the number bank to fill in the chart. Some numbers will satisfy more than one cell, but each number can only be used once.



Source: http://www.transum.org/Software/sw/Starter_of_the_day/Students/Satisfy.asp?Level=2

3. What does $\sqrt{16}$ equal?

4. What does $\sqrt{36}$ equal?

5. What does 4×4 equal?

6. What does 6×6 equal?

7. Does $\sqrt{16} = \sqrt{4 \times 4}$?

8. Does $\sqrt{36} = \sqrt{6 \times 6}$?

9. Rewrite $\sqrt{20}$ using at least one perfect square factor.

10. Rewrite $\sqrt{28}$ using at least one perfect square factor.

Simplify the square roots as much as possible.

11. $\sqrt{18}$

12. $\sqrt{44}$

13. $\sqrt{169}$

14. $\sqrt{75}$

15. $\sqrt{128}$

16. $\sqrt{250}$

17. $\sqrt{12}$

18. $\sqrt{144}$

19. $\sqrt{98}$

20. $\sqrt{343}$

21. $\sqrt{72ab^2c^4}$

22. $\sqrt{175w^4v^3}$

23. $\sqrt{80a^3}$

24. $\sqrt{256}$

25. $\sqrt{56x^3y^2}$

26. $\sqrt{16}$

27. $\sqrt{200xyz^2}$

28. $\sqrt{45}$

29. Josue simplified $\sqrt{450}$ as $15\sqrt{2}$. Is he correct? Explain why or why not.

30. Tiah was absent from school the day that you learned how to simplify a square root. Using $\sqrt{360}$, write Tiah an explanation for simplifying square roots.

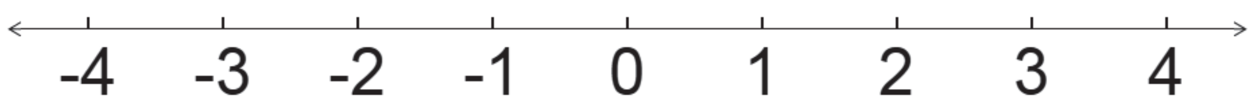
31. Simplify $\sqrt{100 \cdot 9 \cdot 16}$

32. Simplify $\sqrt[3]{8 \cdot (-1) \cdot 27}$

33. Simplify $\sqrt{45x^2y}$

34. Simplify $\sqrt[3]{54x^3y^4}$

35. A. Place the numbers $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$ on the number line below.



B. Place the numbers $\sqrt{2}$ and $\sqrt{3}$ on the number line. Between what two integers do these numbers fall?

C. How could we represent -3 with a radical? Is there any other way to represent -3 with a radical?

For Problems 36–44, determine if each equation is True or False.

36. $\sqrt{64} = 2^3$

37. $\sqrt{8} = 2^{\frac{3}{2}}$

38. $\sqrt{50} = 25\sqrt{2}$

39. $4^{\frac{3}{4}} = 64^{\frac{1}{4}}$

40. $6^{\frac{3}{2}} = 6\sqrt{6}$

41. $\sqrt{98} = 7 \cdot 2^{\frac{1}{2}}$

42. $16^{\frac{3}{8}} = 2 \cdot 2^{\frac{1}{2}}$

43. $\sqrt[3]{125} = \sqrt{25}$

44. $\sqrt{200} = 10 \cdot 2^{\frac{1}{2}}$

Simplify each expression.

45. $\sqrt[3]{250a^5}$

46. $\sqrt[3]{135m^8}$

47. $\sqrt[3]{-448n^2}$

48. $\sqrt[3]{-40x^4}$

49. $2\sqrt[3]{320n^8}$

50. $-3\sqrt[3]{-216x^3}$

51. $\sqrt[5]{64x^6y^8}$

52. $\sqrt[4]{1250a^3b^8}$

53. $2\sqrt[4]{768a^7b^5}$

54. $\sqrt[5]{96n^3m^5}$

55. $-\sqrt[3]{-40a^3b^6}$

56. $\sqrt[5]{x^{25}y^{17}z^3}$