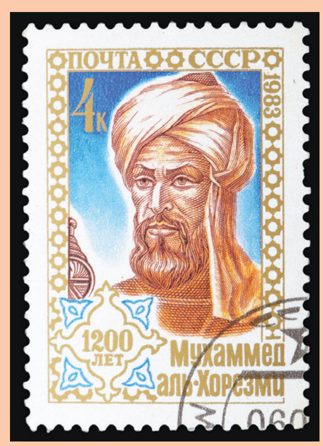
LESSON 21 Completing the Square Like Ancient Mathematicians

LEARNING OBJECTIVES

- Today I am: reading about how the mathematician, al-Khwarizmi, solved a quadratic equation using an area model.
- So that I can: learn to complete the square.
- I'll know I have it when I can: solve al-Khwarizmi's problem.

Opening Reading

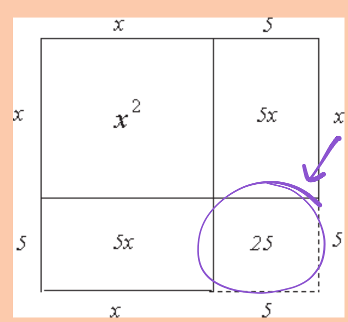
One of the earliest of the Arab mathematicians, Muhammad ibn Musa al-Khwarizmi (approximately 780–850 CE), was employed as a scholar at the House of Wisdom in Baghdad in present day Iraq. Al-Khwarizmi wrote a book on the subjects of *al-jabr* and *al-muqabala*. Al-Khwarizmi's word *al-jabr* eventually became our word *algebra*, and, of course, the subject of his book was what we call algebra. In his algebra book, al-Khwarizmi solves the following problem:



What must be the square which, when increased by ten of its own roots, amounts to 39?

The equation al-Khwarizmi wanted to solve is $x^2 + 10x = 39$.

When al-Khwarizmi solved the equation $x^2 + 10x = 39$ by completing the square, he completed an actual square. The solid line portion of the figure has area $x^2 + 5x + 5x = x^2 + 10x$.

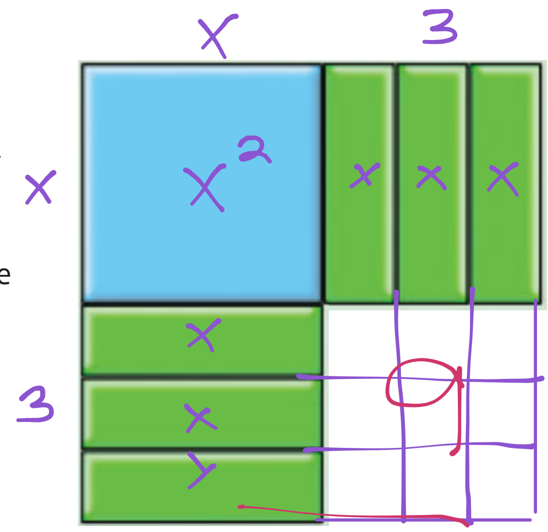


1. Discuss with your group, how al-Khwarizmi may have solved this problem. How did the square he drew help him solve this problem?

Let's see how al-Khwarizmi may have solved his quadratic equation by looking at some other examples with algebra tiles.

Completing the Square Investigation

2. At the right, is a partial square representing $x^2 + 6x$.



   A. Mark each rectangle with the algebraic expression it represents.

   B. How many unit tiles would you need to complete the square?

   9 unit tiles

   C. What are the dimensions of the completed square?

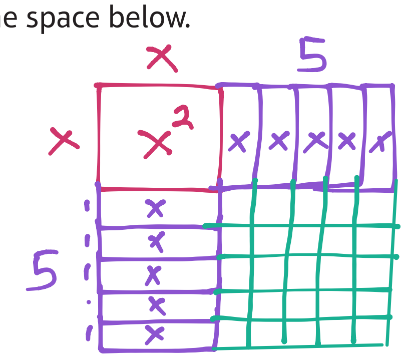
   $(x+3)(x+3)$

   D. Replace c and the question mark to make the statement true.

   $x^2 + 6x + \overset{9}{c} = (x + \overset{3}{?})^2$ $\qquad c =$ 9 $\qquad ? =$ 3

   $x^2+6x+9 = (x+3)^2$

3. A. Draw a partial square with algebra tiles to represent $x^2 + 10x$ in the space below.



   B. How many unit tiles would you need to complete the square?

   25 unit tiles

   C. What are the dimensions of the completed square?

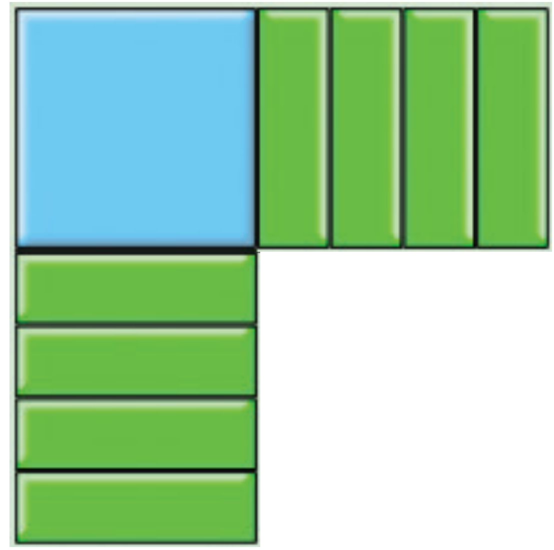
   $(x+5)(x+5)$

   D. Replace c and the question mark to make the statement true.

   $x^2 + 10x + c = (x + ?)^2$ $\qquad c =$ ______ $\qquad ? =$ ______

   $x^2+10x+25 = (x+5)^2$

4. At the right, is a partial square representing $x^2 + 8x$.



   A. Mark each rectangle with the algebraic expression it represents.

   B. How many unit tiles would you need to complete the square?

   16

   C. What are the dimensions of the completed square?

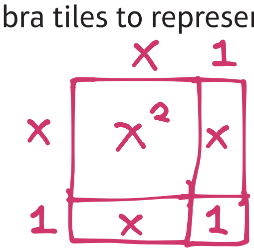
   $(x+4)(x+4)$

   D. Replace c and the question mark to make the statement true.

   $x^2 + 8x + c = (x + ?)^2$ $\qquad c =$ ______ $\qquad ? =$ ______

   $x^2 + 8x + \underline{16} = (x + \underline{4})^2$

5. A. Draw a partial square with algebra tiles to represent $x^2 + 2x$ in the space below.



   B. How many unit tiles would you need to complete the square?

   1 unit tile

   C. What are the dimensions of the completed square?

   $(x+1)(x+1)$

   D. Replace c and the question mark to make the statement true.

   $x^2 + 2x + c = (x + ?)^2$ $\qquad c =$ ______ $\qquad ? =$ ______

   $x^2 + 2x + \underline{1} = (x + \underline{1})^2$

$x^2 + 6x + \underline{9} = (x + \underline{3})^2$

$x^2 + 10x + \underline{25} = (x + \underline{5})^2$

$x^2 + 8x + \underline{16} = (x + \underline{4})^2$

$x^2 + 2x + \underline{1} = (x + \underline{1})^2$

Discussion

6. A. How do you determine the c in each of these cases?

$\left(\frac{b}{2}\right)^2 \to c$

B. How do you determine the ? in each of these cases?

$\frac{b}{2} \to ?$

7. In the expression, $x^2 + bx + c$, how do you use b to get the value of c to form a perfect square?

$\left(\frac{b}{2}\right)^2 = c$

Practice Exercises

Find the missing c in each problem and then rewrite the trinomial as a perfect square binomial.

8. $x^2 + 12x + c$ $\left(\frac{12}{2}\right)^2$

$x^2 + 12x + \underline{36}$

$(x+6)^2$

9. $x^2 + 20x + c$ $\left(\frac{20}{2}\right)^2$

$x^2 + 20x + \underline{100}$

$(x+10)^2$

10. $x^2 - 4x + c$ $\left(\frac{-4}{2}\right)^2$

$x^2 - 4x + \underline{4}$

$(x-2)^2$

11. $x^2 - 6x + c$

$x^2 - 6x + 9$

$(x-3)^2$

12. $x^2 - 10x + c$

$x^2 - 10x + 25$

$(x-5)^2$

13. $x^2 - 12x + c$

$x^2 - 12x + 36$

$(x-6)^2$

14. $x^2 + 3x + c$ $\left(\frac{3}{2}\right)^2$

$x^2 + 3x + \frac{9}{4}$

$\left(x + \frac{3}{2}\right)^2$

15. $x^2 - 3x + c$

$x^2 - 3x + \frac{9}{4}$

$\left(x - \frac{3}{2}\right)^2$

16. $x^2 + 7x + c$ $\left(\frac{7}{2}\right)^2$

$x^2 + 7x + \frac{49}{4}$

$\left(x + \frac{7}{2}\right)^2$

17. $x^2 + bx + c$

$x^2 + bx + \left(\frac{b}{2}\right)^2$

$\left(x + \frac{b}{2}\right)^2$

18. al-Khwarizmi's equation was $x^2 + 10x = 39$. Let's look at each side of his equation.

Steps	$\left(\frac{10}{2}\right)^2$ **Algebra Work**
A. Complete the square on the left side of the equation. How many units did you need to add?	$x^2 + 10x +$ 25 $= 39 +$ 25
B. Add the same amount to the right side.	
C. What is your new equation?	$\sqrt{(x+5)^2} = \pm\sqrt{64}$
D. Take the square root of both sides. Don't forget the ± sign.	$x+5 = \pm 8$ $-5 \quad -5$
E. What are the answers to al-Khwarizmi's equation?	$x = -5 \pm 8$ $-5+8, \; -5-8$
F. Check to see if both numbers make the original statement true.	$x = 3, \; -13$

19. al-Khwarizmi gave his instructions for solving the problem in words rather than symbols, as follows:

> *What must be the square which, when increased by ten of its own roots, amounts to 39? The solution is this: You have the number of roots, which in the present instance yields five. This you multiply by itself; the product is 25. Add this to 39; the sum is 64. Now take the root of this which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for.*

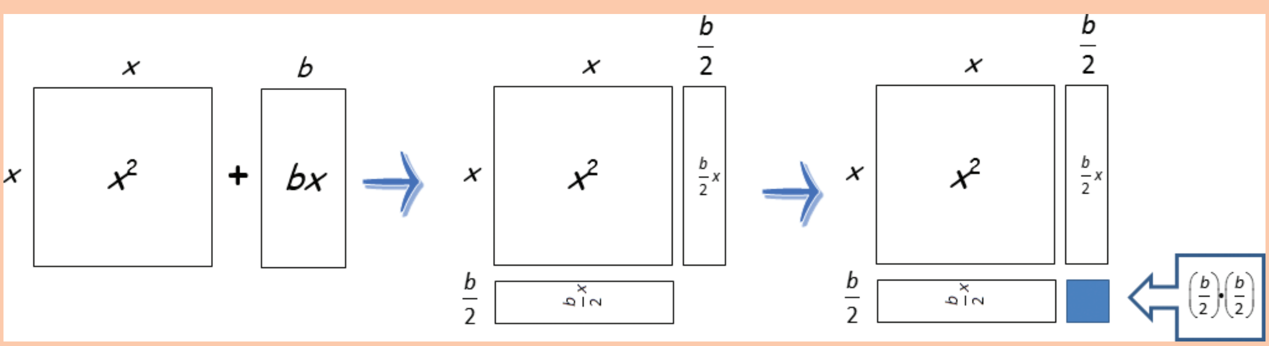
How does al-Khwarizmi's solution compare to the process you used in Exercise 18?

Discussion

20. Using the patterns developed in this lesson, how could you factor the expression, $x^2 - bx + c$.

21. Explain what is shown in each stage of completing the square in the Lesson Summary.

Lesson Summary



NAME: ____________________ PERIOD: ____________ DATE: ____________

Homework Problem Set

Find the missing *c* in each problem and then rewrite the trinomial as a perfect square binomial.

1. $x^2 + 24x + c$

2. $x^2 + 28x + c$

3. $x^2 - 36x + c$

4. $x^2 - 70x + c$

5. $x^2 - 20x + c$

6. $x^2 - 24x + c$

7. $x^2 + 1x + c$

8. $x^2 - 1x + c$

9. $x^2 + 5x + c$

10. $x^2 + 9x + c$

Spiral REVIEW—Simplifying Radicals

Simplify each radical expression.

11. $\sqrt{12}$

$\sqrt{4} \cdot \sqrt{3}$

$2\sqrt{3}$

12. $\sqrt{18}$

$\sqrt{9}\ \sqrt{2}$

$3\sqrt{2}$

13. $\sqrt{24}$

14. $\sqrt{7}$

15. $\sqrt{36}$

16. $\sqrt{50}$

17. $\sqrt{20}$

18. $\sqrt{5}$

Spiral REVIEW—Solving Equations

Solve each equation.

19. $27 = -3 + 5(x + 6)$

20. $-13 = 5(2 + 4m) - 2m$

21. $4(-x + 4) = 12$

22. $-2 = -(n - 8)$

23. $-6(1 - 5v) = 54$

24. $8 = 8v - 4(v + 8)$

25. $10(1 + 3b) = -20$

26. $-5n - 8(1 + 7n) = -8$

27. $8(4k - 4) = -5k - 32$